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A tunneling picture of dual giant Wilson loop

Akitsugu MIWA^{a1}, Yoske SUMITOMO^{b2} and Kentaroh YOSHIDA^{c3}

^a *Institute of Physics, University of Tokyo
Komaba, Meguro-ku, Tokyo 153-8902, Japan*

^b *Department of Particle and Nuclear Physics,
The Graduate University for Advanced Studies,
Tsukuba, Ibaraki 305-0801, Japan.*

^c *Kavli Institute for Theoretical Physics,
University of California,
Santa Barbara, CA. 93106, USA.*

Abstract

We further discuss a rotating dual giant Wilson loop (D3-brane) solution constructed in Lorentzian AdS by Drukker et al. The solution is shown to be composed of a dual giant Wilson loop and a dual giant graviton by minutely examining its shape. This observation suggests that the corresponding gauge-theory operator should be a k -th symmetric Wilson loop with the insertions of dual giant graviton operators. To support the correspondence, the classical action of the solution should be computed and compared with the gauge-theory result. For this purpose we first perform a Wick rotation to the Lorentzian solution by following the tunneling prescription and obtain Euclidean solutions corresponding to a circular or a straight-line Wilson loop. In Euclidean signature boundary terms can be properly considered in the standard manner and the classical action for the Euclidean solutions can be evaluated. The result indeed reproduces the expectation value of the k -th symmetric Wilson loop as well as the power-law behavior of the correlation function of dual giant graviton operators.

¹e-mail address: akitsugu@hep1.c.u-tokyo.ac.jp

²e-mail address: sumitomo@post.kek.jp

³e-mail address: kyoshida@kitp.ucsb.edu

1 Introduction

One of the long-standing ideas in particle physics is to make a connection between a Wilson loop in gauge theory and a string-like object like in string theory. In the context of AdS/CFT correspondence [1], it is proposed that the expectation value of the fundamental Wilson loop is given by the “area law” of the fundamental string world-sheet attached to the loop on the AdS boundary [2, 3]. For straight lines and circular loops, the area of the string world-sheet is shown to reproduce the expectation value of the Wilson loop calculated by summing up the planar ladder diagrams in a large ’t Hooft coupling limit ($\lambda \equiv Ng_{\text{YM}}^2 \rightarrow \infty$).

One may consider a multiply wrapped Wilson loop or a Wilson loop in higher-dimensional representation [4]. It can carry a multiple winding number, say k , in terms of the fundamental representation. Hence a natural candidate for its counterpart is a state with string charge k . The multi-string state can be described as a spike D-brane solution with non-trivial electric flux describing the string charge [5]. It is now proposed that an anti-symmetric representation corresponds to an $\text{AdS}_2 \times \text{S}^4$ D5-brane [6] called “giant Wilson loop,” and a symmetric representation to an $\text{AdS}_2 \times \text{S}^2$ D3-brane [4, 7, 8] called “dual giant Wilson loop.” The names are analogy to (dual) giant gravitons [9–11].

With the help of the string charge k , it is possible to consider a new double-scaling limit, which is different from the usual large N limit. In the case of k -th symmetric representation, k and N are taken to be large while keeping $\kappa \equiv k\sqrt{\lambda}/4N$ fixed. The expectation value of the Wilson loop can be evaluated by using a Gaussian matrix model and the result completely agrees with the classical action of a D3-brane solution in the above limit [4].⁴ Note that the classical action contains non-planar contributions in spite of large N , because large k fundamental strings are bound on the D3-brane.

As another generalization, an R-charge J may be introduced in analogy with [18]. A string solution rotating in S^5 has been constructed as a counterpart of a fundamental Wilson loop with local operator insertions Z^J and its Hermitian conjugate [19]. Here Z is a complex scalar field in $\mathcal{N}=4$ SYM and related to a $\text{U}(1)$ R-charge. Then an open spin chain description was discussed.

Remember that the expectation values of Wilson loops are usually discussed in Euclidean signature. Euclidean AdS is important also from the viewpoint of the bulk-boundary correspondence for local operators with R-charge. When in Lorentzian signature, the classical solutions that correspond to such operators are introduced at the center of AdS and do not

⁴A symmetric Wilson loop cannot be distinguished from a multiply wrapped one in the leading-order of approximation at strong coupling [12, 13]. See [14] for an argument on a sub-leading contribution in the string side. The approach based on Gaussian matrix model was argued in [15, 16] and also in a recent work [17].

reach the boundary. That is why a double Wick rotation has to be performed by following [20]. Then the bulk-boundary correspondence can be discussed by using the semi-classical bulk modes propagating along the “tunneling trajectory,” connecting the two points on the boundary.

The tunneling method is also applicable to the fundamental Wilson loop with local operator insertions [21, 22].⁵ The double Wick rotation for the Lorentzian solution [19] leads to a Euclidean string solution which attaches to the Wilson loop on the boundary and propagates along the tunneling trajectory. Its classical action certainly reproduces the expectation value of the corresponding Wilson loop.

In this paper we discuss a k -th symmetric Wilson loop carrying an R-charge. Then a D3-brane solution rotating in S^5 may be discussed as the string-theory counterpart (i.e., a rotating dual giant Wilson loop). In fact, a rotating D3-brane solution has already been constructed in Lorentzian signature [25]. Here we investigate the shape of the solution in detail. Then the solution is shown to be composed of a dual giant Wilson loop and a dual giant graviton [10, 11], rather than a rotating BPS particle. Thus this observation suggests that the dual gauge-theory operator should be a k -th symmetric Wilson loop with the insertions of dual giant graviton operators [26, 27], rather than Z^J .

Next we construct a Euclidean solution by applying the double Wick rotation for the Lorentzian solution. Then its classical action is evaluated by properly taking account of boundary terms. Although the computation is complicated the result is simple; The resulting action reproduces the expectation value of the k -th symmetric Wilson loop and also a two point function of the local operators with R-charge J as it is expected.

This paper is organized as follows: Section 2 is a brief review of the tunneling picture. Its new application to a dual giant graviton is also discussed. Section 3 is also a review of the rotating D3-brane solution constructed in [25]. We newly find the relation between the solution and a dual giant graviton. This is the key observation to correctly identify the corresponding gauge-theory operator. In subsections 3.2 a double Wick rotation is performed to the Lorentzian solution by following the tunneling prescription. The resulting Euclidean solution attaches to a circle or a straight line on the boundary and propagates along the tunneling trajectory. In section 4 the classical action of the Euclidean solution is evaluated. Then in section 5 we discuss the relation between the resulting action and the expectation value of the Wilson loop. Section 6 is devoted to a conclusion and discussions.

⁵A two-spin string around the tunneling trajectory is also discussed in [23], and also a related work has been done in [24].

2 Tunneling picture of bulk-boundary correspondence

The bulk-boundary correspondence can be manifestly discussed in Euclidean formulation. For this purpose the Wick rotation should be performed. But note that we are interested in the case with an angular momentum, where a subtlety for the Wick rotation exists [20]. Then the tunneling prescription should be utilized. It would be available for later discussion to give a brief review of the tunneling prescription with the three examples: 1) a BPS particle (BMN case), 2) a dual giant graviton, 3) a rotating string world-sheet. Note that the cases 1) and 3) are just reviews of the preceding works, but the case 2) has not been discussed in the earlier literatures and this is the first attempt.

2.1 Tunneling trajectory of BPS particle

We give a brief review of the tunneling prescription by taking a BPS particle rotating in S^5 with an angular momentum J . Here we assume that J is much less than N .

The $AdS_5 \times S^5$ geometry in global coordinates is given by

$$\frac{ds^2}{L^2} = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho (d\chi^2 + \sin^2 \chi (d\varphi_1^2 + \sin^2 \varphi_1 d\varphi_2^2)) + d\theta^2 + \sin^2 \theta d\phi^2, \quad (2.1)$$

$$\frac{C_4}{L^4} = \sinh^4 \rho \sin^2 \chi \sin \varphi_1 dt \wedge d\chi \wedge d\varphi_1 \wedge d\varphi_2, \quad (2.2)$$

with a constant dilaton field. The S^2 metric in S^5 is explicitly written down, since we consider classical solutions which are localized with respect to the remaining directions. Thus the S^5 part of the RR potential C_4 is also irrelevant for the solutions.

A null trajectory of a point particle rotating in S^5 is given by

$$\rho = 0, \quad \theta = \frac{\pi}{2}, \quad \phi = t. \quad (2.3)$$

It is known that the string modes propagating along the trajectory correspond to local operators with large R-charge [18]. But the trajectory does not reach the boundary and hence it is not available to discuss the correlation functions of the operators.

A solution for this issue was proposed in [20] and it is based on a semi-classical tunneling phenomenon. Hence the prescription is called “tunneling picture.” From now on let us see the tunneling picture. First the trajectory (2.3) should be recaptured with the Poincaré coordinates of the AdS_5 geometry, in which the $AdS_5 \times S^5$ metric becomes

$$\frac{ds^2}{L^2} = \frac{dZ^2 - (dX_0)^2 + (dX_1)^2 + (dX_2)^2 + (dX_3)^2}{Z^2} + d\theta^2 + \sin^2 \theta d\phi^2. \quad (2.4)$$

Then the trajectory (2.3) is mapped to the following trajectory:

$$Z = \frac{\ell}{\cos t}, \quad X_0 = \ell \tan t, \quad X_1 = X_2 = X_3 = 0, \quad \phi = t. \quad (2.5)$$

Here we have introduced a constant parameter ℓ , which is related to the scale invariance of the metric (2.4). The trajectory (2.5) satisfies the null condition, $Z^{-2}(\dot{Z}^2 - \dot{X}_0^2) + \dot{\phi}^2 = 0$, and also the equations of motion $\dot{X}_0/Z^2 = 1/\ell$ and $\dot{\phi} = 1$.

Here let us see the motion of Z . The null condition and the equations of motion of X_0 and ϕ lead to

$$\dot{Z}^2 + V(Z) = 0, \quad V(Z) \equiv -Z^4/\ell^2 + Z^2.$$

This equation suggests that the classical solution (2.5) does not reach the boundary $Z = 0$ because of the potential barrier coming from $V(Z)$. Thus the trajectory that reaches the boundary is realized as a trajectory that tunnels the potential barrier.

Such a tunneling trajectory was proposed in [20] via the Wick rotation with respect to the parameter t as $t_E = it$ as well as the target space time coordinate as $X_4 = iX_0$. We have to consider simultaneously whether the imaginary angular velocity or equivalently the Wick rotation with respect to the angular direction as $\phi_E = i\phi$ and use the ansatz $\phi_E = t_E$. The resulting tunneling trajectory is given by

$$Z = \frac{\ell}{\cosh t_E}, \quad X_1 = X_2 = X_3 = 0, \quad X_4 = \ell \tanh t_E. \quad (2.6)$$

This describes a semi-circle $Z^2 + X_4^2 = \ell^2$ in the (Z, X_4) plane.

By considering modes propagating along the tunneling trajectory (2.6), we can discuss a correlation function of the local operators with R-charge. The simplest example would be the two point function of the BPS operators:

$$\langle \text{Tr}_F Z^J(\vec{X}_i) \text{Tr}_F \bar{Z}^J(\vec{X}_f) \rangle. \quad (2.7)$$

Here the complex scalar field Z is defined as $\Phi_5 + i\Phi_6$. The points $\vec{X}_{i,f}$ correspond to the two end points, $(X_1, X_2, X_3, X_4) = (0, 0, 0, \pm\ell)$, of the tunneling trajectory (2.6).

Another derivation of tunneling null geodesic

There is another derivation of the tunneling null geodesic (2.6). It should take the following steps:

1. First let us consider the Euclidean AdS by performing the double Wick rotation: $t_E = it$ and $\phi_E = i\phi$.

2. Next we turn to the Euclidean Poincaré coordinates via

$$\begin{aligned} Z &= \frac{\ell}{f}, & X_1 &= \frac{\ell}{f} \sinh \rho \sin \chi \sin \varphi_1 \cos \varphi_2, & X_2 &= \frac{\ell}{f} \sinh \rho \sin \chi \sin \varphi_1 \sin \varphi_2, \\ X_3 &= \frac{\ell}{f} \sinh \rho (\alpha \sin \chi \cos \varphi_1 - \sqrt{1 - \alpha^2} \cos \chi), & X_4 &= \frac{\ell}{f} \sinh t_E \cosh \rho, \\ f &= \cosh t_E \cosh \rho + \sinh \rho (\sqrt{1 - \alpha^2} \sin \chi \cos \varphi_1 + \alpha \cos \chi), \end{aligned} \quad (2.8)$$

$(0 \leq \alpha \leq 1).$

Here a constant parameter α is contained as well as ℓ . It will be related to the shape of the Wilson loop later. The transformation (2.8) can be decomposed into a series of simple coordinate transformations as explained in Appendix A.

After the double Wick rotation in the first step, the null trajectory (2.3) has been mapped to the trajectory described by $\rho = 0$, $\theta = \pi/2$ and $\phi_E = t_E$. Then by performing the transformation (2.8), it is mapped to the tunneling trajectory (2.6).

In subsection 2.2 the above steps are applied to a dual giant graviton solution [10,11], which corresponds to a local operator carrying an R-charge of order N or larger. In subsection 2.3 we give a brief review of the Euclidean string solution of [21,22], which was constructed by applying the above steps to the Lorentzian solution of [19]. In the next section we apply the above steps to a dual giant Wilson loop (D3-brane) solution rotating in S^5 [25].

2.2 Dual giant graviton around tunneling trajectory

From now on let us discuss the tunneling picture of a dual giant graviton solution. Here we assume that its angular momentum (R-charge) is the same order as N or larger.

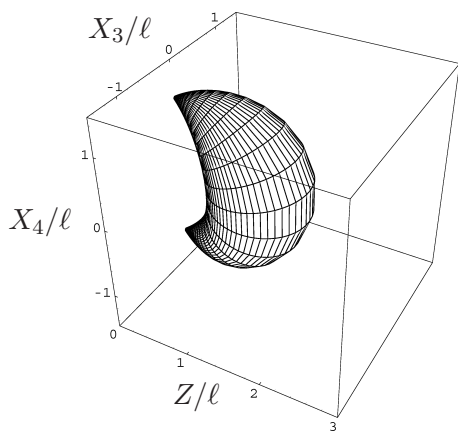
The coordinates t , χ , φ_1 and φ_2 are used as the world-volume coordinates of the dual giant graviton solution. Then the solution will be given by

$$\rho = \bar{\rho} : \text{const.}, \quad \theta = \frac{\pi}{2}, \quad \phi = t. \quad (2.9)$$

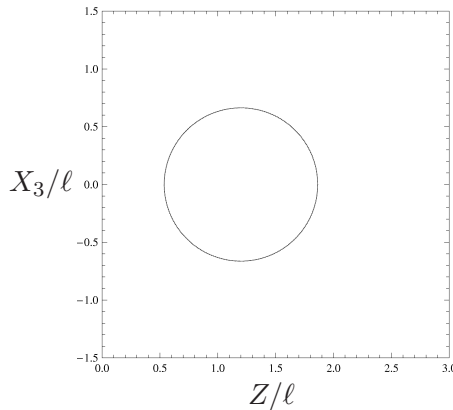
The double Wick rotation just changes the last equation of (2.9) to $\phi_E = t_E$.

Next we perform the transformation (2.8). For simplicity the case with $\alpha = 0$ is discussed. Let us first concentrate on the slice of the solution on which the relation $\sin \varphi_1 = 0$ is satisfied. This slice corresponds to the north and the south pole of the S^2 spanned by φ_1 and φ_2 . Then the remaining directions of the world-volume are two-dimensional. Indeed, for $\sin \varphi_1 = 0$, X_1 and X_2 vanish and the solution in terms of (Z, X_3, X_4) is given by a two-dimensional surface depicted in Fig. 1-(a).

Figure 1-(b) is the cross section of Fig. 1-(a) at $t = 0$, i.e., (Z, X_3) -plane at $X_4 = 0$. The tunneling trajectory penetrates the plane and the point is located at $(Z, X_3) = (\ell, 0)$. As far



(a)



(b)

Figure 1: Euclidean dual giant graviton. Its shape is depicted in (a). A time slice at $t = 0$ of (a) is plotted in (b).

as Z , X_3 and X_4 are concerned, all other points on the D3-brane, i.e., the region $0 < \varphi_1 < \pi$ are contained inside the surface of Fig. 1-(a). As for X_1 and X_2 , points on the solution are in the region: $X_1^2 + X_2^2 \leq \ell^2 \sinh^2 \bar{\rho}$.

The propagation of the dual giant graviton should correspond to a two point function of the dual giant graviton operators [26, 27]:

$$\langle \text{Tr}_{S_J} Z(\vec{X}_i) \text{Tr}_{S_J} \bar{Z}(\vec{X}_f) \rangle. \quad (2.10)$$

Here the trace is taken over the J -th symmetric representation. So far we have assumed that J is the same order as N or larger, but it may be possible to consider the limit $J \ll N$ in (2.10). Then all of non-planar contributions in the dual giant graviton operator are negligible. After all, (2.10) is reduced to (2.7). In the bulk gravity side a dual giant graviton shrinks in the same limit and it should be regarded as a BPS particle.

2.3 String world-sheet around tunneling trajectory

Finally let us remember the tunneling picture of a rotating string world-sheet [21, 22].

We shall begin with the Lorentzian solution [19]. Taking t and ρ as world-sheet coordinates, it is given by

$$\chi = 0, \pi, \quad \varphi_1 = \text{const.}, \quad \varphi_2 = \text{const.}, \quad \sin \theta = \frac{1}{\cosh \rho}, \quad \phi = t. \quad (2.11)$$

Two patches, $\chi = 0$ and $\chi = \pi$, are attached to straight lines on the boundary at $\rho = \infty$ and they are sewn together at $\rho = 0$. The solution carries an angular momentum from the infinite past $t = -\infty$ to the infinite future $t = \infty$.

As proposed in [19], a natural candidate for the dual gauge-theory operator would be the Wilson loop operator with local operator insertions⁶:

$$W_{Z^J} \equiv \text{Tr}_F \mathcal{P} \left[Z^J(t = -\infty) e^{i \int_{-\infty}^{\infty} dt (A_t + \Phi_4)} \bar{Z}^J(t = \infty) e^{i \int_{\infty}^{-\infty} dt (A_t + \Phi_4)} \right]. \quad (2.12)$$

This operator contains two Wilson lines extending from $t = -\infty$ to $t = \infty$. Each of them corresponds to the line given by $(\rho, \chi) = (\infty, 0)$ and (∞, π) on which the string world-sheet is attached. The local operators Z^J and \bar{Z}^J may be regarded as a “creation” and an “annihilation” operator of the R-charge, respectively. The R-charge “created” by Z^J at the infinite past is carried by the rotating string to the infinite future and then it is “annihilated” by \bar{Z}^J .

Although it is interesting proposal, this Lorentzian picture is not available when calculating the expectation value of the operator via the classical string action. This is because the angular momentum is carried from the infinite past to the infinite future, and it does not reach the boundary. As a result, the operator insertions must be assumed at the infinite past and future. The situation is just the same as in the case of correlation functions of local operators with R-charge. In fact, by applying the steps introduced in subsection 2.1, we can construct a solution corresponding to a Wilson loop with the insertions of local operators in a finite region on the AdS boundary [21, 22].

After performing the steps 1. and 2. in subsection 2.1 to the solution (2.11), the AdS₅ part of the resulting solution is given by

$$\begin{aligned} Z &= \frac{\ell}{\cosh t_E \cosh \rho \pm \alpha \sinh \rho}, \quad X_1 = X_2 = 0, \\ X_3 &= \frac{\mp \ell \sqrt{1 - \alpha^2} \sinh \rho}{\cosh t_E \cosh \rho \pm \alpha \sinh \rho}, \quad X_4 = \frac{\ell \sinh t_E \cosh \rho}{\cosh t_E \cosh \rho \pm \alpha \sinh \rho}. \end{aligned} \quad (2.13)$$

This is the string solution constructed in [21, 22]. Figures 2-(a), (b) and (c) depict the solutions with $\alpha = 0, 0.7$ and 1.0 , respectively.

By setting $\rho = 0$ in (2.13), the solution contains the tunneling trajectory (2.6). In fact, the solution carries angular momentum from the one end point \vec{X}_i of the tunneling trajectory (2.6) to the other end point \vec{X}_f .

On the other hand, by taking large ρ limit, the string world-sheet is attached to a circle ($\alpha \neq 1$) or a straight line ($\alpha = 1$) on the AdS boundary $Z = 0$.

It was shown in [22] that the action of the string solution reproduces correct ℓ - and α -dependences of the expectation values of the following operator:

$$\left\langle \text{Tr}_F \mathcal{P} \left[\exp \left(\oint_C ds \left[i A_\mu(\vec{X}(s)) \dot{X}^\mu(s) + \sqrt{\dot{\vec{X}}^2(s)} \Phi^4(\vec{X}(s)) \right] \right) Z^J(\vec{X}_i) \bar{Z}^J(\vec{X}_f) \right] \right\rangle. \quad (2.14)$$

⁶See also the explanation in [25].

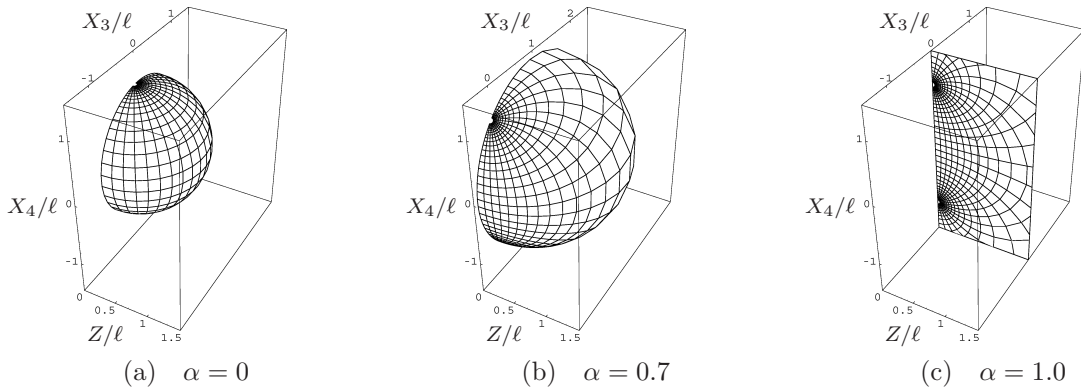


Figure 2: Examples of string world-sheet attached to a circle or a straight line. Figures (a), (b) and (c) correspond to the case with $\alpha = 0, 0.7$ and 1.0 , respectively. The world-sheet contains tunneling trajectory and carries an angular momentum along it. Here t and θ are used as parameters, while t and $\sigma \equiv \text{arctanh}(\sin \theta)$ are used in [22].

The shape of the loop C is the same as that of the boundary of the solution (2.13). Note that both of \vec{X}_i and \vec{X}_f are located on the loop C .

Here the trace is taken over the fundamental representation. The aim of this paper is to extend the analysis to a k -th symmetric Wilson loop with local operator insertions. In the next section we consider the tunneling picture of a rotating dual giant Wilson loop (D3-brane) solution.

3 Tunneling picture of dual giant Wilson loop

In this section we discuss a tunneling picture of dual giant Wilson loop. That is, the rotating string solution in subsection 2.3 is extended to a rotating D3-brane solution.

We first reexamine the rotating D3-brane solution in Lorentzian AdS [25]. It is a generalization of the string solution (2.11) to the D3-brane case. The shape of the solution leads us to observe that it is composed of a dual giant Wilson loop and a dual giant graviton.

Then we have to perform the double Wick rotation and the coordinate transformation in subsection 2.1. After that, the resulting solution is attached to a circle or straight line on the boundary of Euclidean Poincaré AdS and carrying an angular momentum from a point on the boundary to another.

3.1 Lorentzian solution and its properties

Here we introduce a rotating dual giant Wilson loop (D3-brane) solution constructed in [25].

Let us begin with the global coordinates (2.1). The coordinates t , ρ , φ_1 and φ_2 are regarded as the world-volume coordinates, and the following ansatz is assumed for the region $0 \leq \chi \leq \pi/2$:

$$\chi = \chi(\rho), \quad \theta = \theta(\rho), \quad \phi = t, \quad F_{t\rho} = \frac{L^2}{2\pi\alpha'} F(\rho). \quad (3.1)$$

Here $F_{t\rho}$ is an electric flux induced by smeared string charges. Under this ansatz, the Dirac-Born-Infeld (DBI) action and the Wess-Zumino (WZ) term for the region $0 \leq \chi \leq \pi/2$ are simplified as

$$S_{\text{DBI}} = -\frac{2N}{\pi} \int dt d\rho \sinh^2 \rho \sin^2 \chi \sqrt{(\cosh^2 \rho - \sin^2 \theta)(1 + \sinh^2 \rho \chi'^2 + \theta'^2) - F^2}, \quad (3.2)$$

$$S_{\text{WZ}} = -\frac{2N}{\pi} \int dt d\rho \sinh^4 \rho \sin^2 \chi \chi', \quad (3.3)$$

where we have used the definition of D3-brane tension

$$T_{\text{D3}} \equiv \frac{1}{(2\pi)^3 l_s^4 g_s} = \frac{N}{2\pi^2 L^4}.$$

It is still difficult to find a classical solution even after assuming the ansatz. A sensible way is to require the solution to preserve some supersymmetries. Then it is possible to find a solution by solving BPS equations rather than complicated equations of motion. In fact, the solution concerned here has been derived by requiring a quarter BPS condition [25].

The solution of [25] is given by

$$\sin \chi(\rho) = \frac{C_2 \coth \rho}{\sqrt{\cosh^2 \rho - C_1^2}}, \quad \sin \theta(\rho) = \frac{C_1}{\cosh \rho}, \quad (3.4)$$

$$F(\rho) = -\frac{\cosh^4 \rho - C_1^2}{\cosh^2 \rho \sqrt{\cosh^2 \rho - C_1^2 - C_2^2 \coth^2 \rho}}. \quad (3.5)$$

For the region $\pi/2 \leq \chi \leq \pi$ the following replacement is necessary: $\chi(\rho) \rightarrow \pi - \chi(\rho)$ and $F(\rho) \rightarrow -F(\rho)$.

The two constant parameters C_1 and C_2 are related to two conserved charges, an angular momentum J and a string charge k . The parameter C_2 is related to k through

$$C_2 = \frac{k\sqrt{\lambda}}{4N} \equiv \kappa. \quad (3.6)$$

Thus C_2 is nothing but κ in the notation of [4]. On the other hand, J is given by

$$J = 2 \times \frac{2N}{\pi} C_2 C_1^2 \int d\rho \frac{(\cosh^2 \rho - C_1^2)^2 + C_2^2 \cosh^4 \rho}{(\cosh^2 \rho - C_1^2)^2 \cosh^2 \rho \sqrt{\cosh^2 \rho - C_1^2 - C_2^2 \coth^2 \rho}}. \quad (3.7)$$

The overall factor 2 appears taking into account of the two patches.

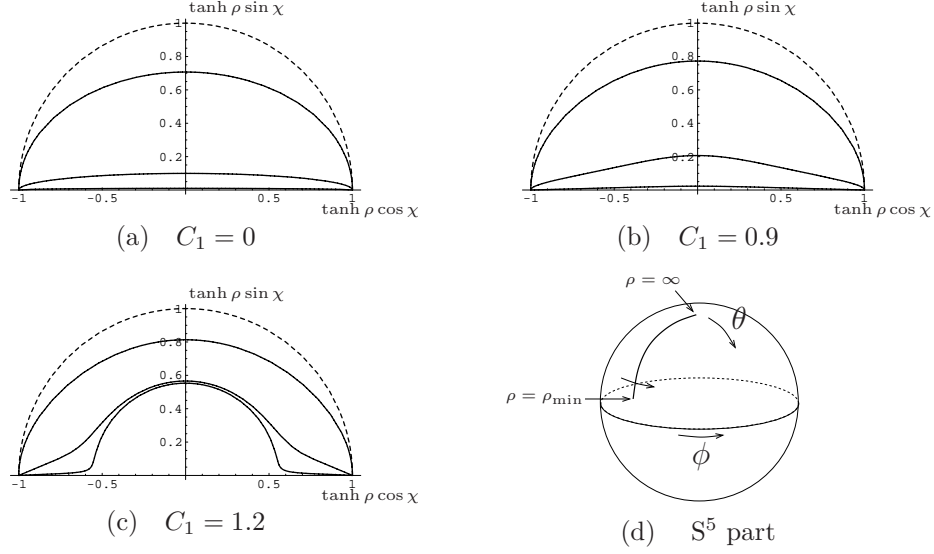


Figure 3: The D3-brane solution in the global coordinates. For various values of C_1 and C_2 it is numerically plotted.

Each D3-brane is attached to the AdS boundary $\rho = \infty$ at $\chi = 0$ and π . Two patches with $0 \leq \chi \leq \pi/2$ and $\pi/2 \leq \chi \leq \pi$ are sewn together smoothly at $\chi = \pi/2$. The radial coordinate ρ takes the minimal value, ρ_{\min} , at $\chi = \pi/2$.⁷

Some typical solutions are numerically plotted in Fig. 3. Figures 3-(a), (b) and (c) show the (ρ, χ) plane with an arbitrary $(t, \varphi_1, \varphi_2)$ for $C_1 = 0, 0.9$ and 1.2 , respectively. In particular, Fig. 3-(a) corresponds to the non-rotating Drukker-Fiol solution [4]. The radial and the angular coordinates of these figures are taken to be $\tanh \rho$ and χ , respectively. Each broken line corresponds to the boundary of the AdS₅ ($\rho = \infty$) and three solid lines in each of the figures show the solutions with $C_2 = 1.0, 0.1$ and 0.01 from the top down.

Figure 3-(d) describes a typical configuration $\theta = \theta(\rho)$ of the solution on S^5 for a fixed t . When $\rho = \infty$, the solution is sitting at the north pole ($\theta = 0$). As ρ decreases, θ increases. At $\rho = \rho_{\min}$, it comes to the turning point, where θ takes its maximal value. It is symmetric with respect to this point. From the ansatz $\phi = t$, it is rotating in S^5 .

Each point on the curves in Figs. 3-(a), (b) and (c) corresponds to a three-dimensional space parametrized by $(t, \varphi_1, \varphi_2)$. Each S^2 parametrized by (φ_1, φ_2) is centered at the horizontal axis. Its radius is given by $L \sinh \rho \sin \chi$ and greater than LC_2 . Hence, when C_2 is kept finite, the radius of the S^2 is much larger than the string length in the large λ limit.

Figure 3-(c) shows a typical behavior of the solution with $C_1 > 1$. For a small value of C_2 , the solution looks like a dual giant graviton with thin spikes sticking out of the north and

⁷For the solution with $\chi(\rho) = 0$, i.e., the solution with $C_1 \leq 1$ and $C_2 \rightarrow 0$, we define $\rho_{\min} = 0$.

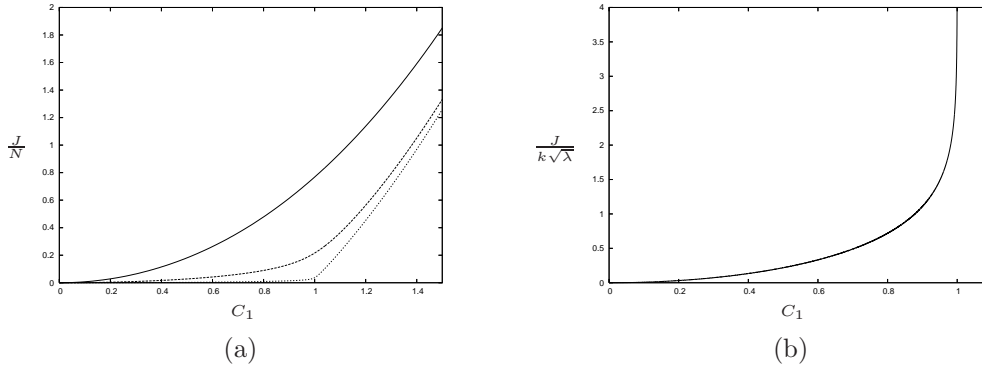


Figure 4: The values of J/N and $J/k\sqrt{\lambda}$ as functions of C_1 .

the south poles. As C_2 increases, the radius of the spike becomes larger and the shape of dual giant graviton tends to be indistinguishable. Also in the case with $C_1 \leq 1$, it is hard to find the dual giant graviton even for small values of C_2 as shown in Figs. 3-(a) and (b).

We shall examine the behavior of J as a function of C_1 for a fixed value of C_2 . Each curve in Fig. 4-(a) shows the behavior of J/N with $C_2 = 1, 0.1$ and 0.01 from the top down. In the limit $C_2 \rightarrow 0$, the curve asymptotically approaches the following line:

$$\frac{J}{N} = 0 \quad (C_1 \leq 1), \quad \frac{J}{N} = C_1^2 - 1 = \sinh^2 \rho_{\min} \quad (C_1 \geq 1). \quad (3.8)$$

The curve $J/N = \sinh^2 \rho_{\min}$ corresponds to the angular momentum of the dual giant graviton whose S^3 -radius is given by $\sinh \rho_{\min} = \sqrt{C_1^2 - 1}$ [10, 11]. Figure 4-(b) depicts the ratio $J/k\sqrt{\lambda}$ in the limit $C_2 \rightarrow 0$.

For finite k the radius of each S^2 becomes much smaller than the string length for the region $\rho > \text{arccosh} C_1$. Hence the analysis with DBI action may not be reliable because of possible α' -corrections. However, a reasonable result has been obtained even for a single string case $k = 1$ and thus the DBI analysis seems to work well even in this case⁸. Now we have no obvious reason to believe it but guess that the corrections cancel each other possibly due to the supersymmetries preserved by the solution.

3.2 Dual giant Wilson loop around tunneling trajectory

Let us now discuss the double Wick rotation and the coordinate transformation (2.8) for the rotating D3-brane solution given by (3.4) and (3.5).

First let us consider the Wick rotation. Here note that an imaginary electric flux $F_{t_E \rho} = -i(L^2/2\pi\alpha')F(\rho)$ should be considered in addition to the double Wick rotation $t_E = it$ and

⁸In particular, by setting $k = 1$ and $C_1 = 1$ the D3-brane solution formally reproduces the string solution in the previous section.

$\phi_E = i\phi$. Then the solution is given by

$$\sin \chi(\rho) = \frac{C_2 \coth \rho}{\sqrt{\cosh^2 \rho - C_1^2}}, \quad \sin \theta(\rho) = \frac{C_1}{\cosh \rho}, \quad \phi_E = t_E, \quad (3.9)$$

$$F(\rho) = -\frac{\cosh^4 \rho - C_1^2}{\cosh^2 \rho \sqrt{\cosh^2 \rho - C_1^2 - C_2^2 \coth^2 \rho}}. \quad (3.10)$$

Next the solution is mapped via (2.8) and then we have the following D3-brane solution in the Euclidean Poincaré coordinate:

$$\begin{aligned} Z &= \frac{\ell}{f}, \quad X_1 = \frac{\ell}{f} \sinh \rho \sin \chi(\rho) \sin \varphi_1 \cos \varphi_2, \quad X_2 = \frac{\ell}{f} \sinh \rho \sin \chi(\rho) \sin \varphi_1 \sin \varphi_2, \\ X_3 &= \frac{\ell}{f} \sinh \rho (\alpha \sin \chi(\rho) \cos \varphi_1 - \sqrt{1 - \alpha^2} \cos \chi(\rho)), \quad X_4 = \frac{\ell}{f} \sinh t_E \cosh \rho, \\ f &= \cosh t_E \cosh \rho + \sinh \rho (\sqrt{1 - \alpha^2} \sin \chi(\rho) \cos \varphi_1 + \alpha \cos \chi(\rho)). \end{aligned} \quad (3.11)$$

Now the function $\chi(\rho)$ is defined by the first equation of (3.9). The shape of the solution is numerically plotted for some values of C_1 and C_2 in Figs. 5 and 6.

In order to see the relation to the Wilson loop, we shall examine the boundary behavior of the solution (3.11) by taking the limit $\rho \rightarrow \infty$. Then, for $\alpha \neq 1$, the boundary of the solution is given by the following trajectory on the AdS boundary:

$$Z = X_1 = X_2 = 0, \quad X_3 = \frac{\mp \ell \sqrt{1 - \alpha^2}}{\cosh t_E \pm \alpha}, \quad X_4 = \frac{\ell \sinh t_E}{\cosh t_E \pm \alpha}. \quad (3.12)$$

The upper (the lower) sign implies the region with $0 \leq \chi \leq \pi/2$ ($\pi/2 \leq \chi \leq \pi$). This is a circle with the radius $\ell/\sqrt{1 - \alpha^2}$ on the (X_3, X_4) -plane. Its center is located at $(X_3, X_4) = (\alpha\ell/\sqrt{1 - \alpha^2}, 0)$.

For $\alpha = 1$, the circle (3.12), except for $t_E = 0$, becomes an infinite line, $X_3 = 0$. That is, the D3-brane is attached to a straight line on the AdS boundary and extended infinitely into the bulk AdS space. This infinitely extended part of the D3-brane can be found by carefully considering $t_E = 0$. For example, let us take the large ρ limit of the solution (3.11) after setting $t_E = 0$. Then we reach the AdS boundary ($Z = 0$) on the patch with $0 \leq \chi \leq \pi/2$ but we go to the region at the vicinity of $Z \sim \infty$ on the other patch $\pi/2 \leq \chi \leq \pi$. It is easy to check that, for $\alpha = 1$, $\sqrt{X_1^2 + X_2^2 + X_3^2}/Z \rightarrow C_2$ in the large ρ limit. This means that the solution asymptotically satisfies the linear ansatz used in [4].

Figures 5 and 6 are some numerical plots of the D3-brane solution with indicated values of C_1 , C_2 and α . Figure 5 depicts two-dimensional surfaces specified by $\sin \varphi_1 = 0$, and Fig. 6 expresses their cross sections at $X_4 = 0$. In particular, Figs. 5-(d) and 6-(d) correspond to the non-rotating Drukker-Fiol solution [4].

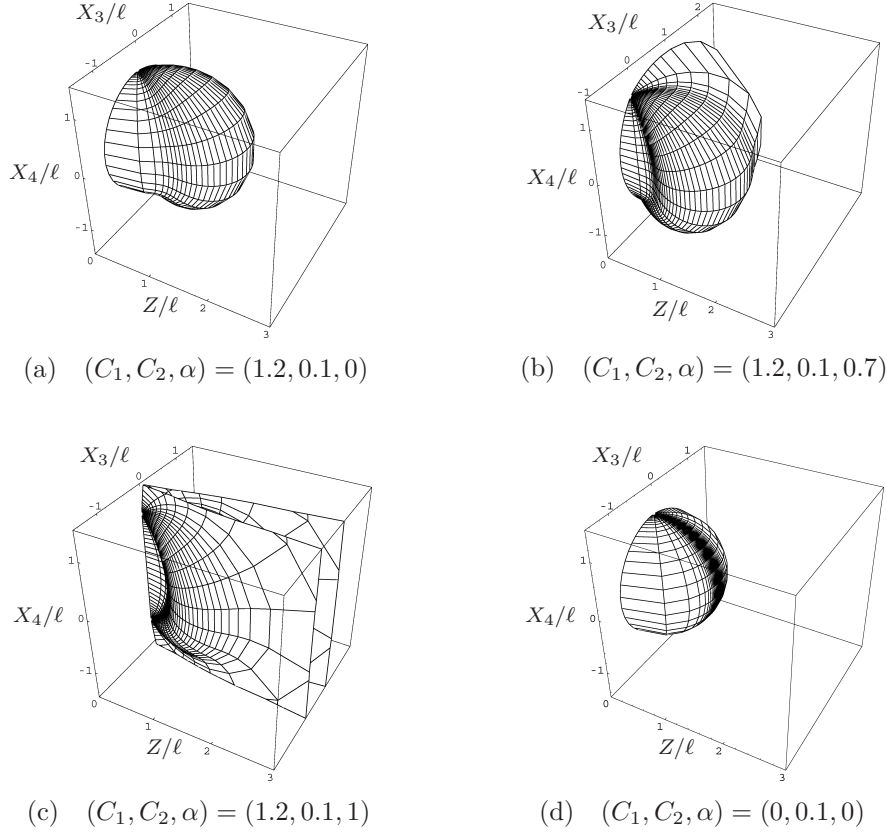


Figure 5: Euclidean D3-brane solution with various values of C_1 , C_2 and α . Figure (d) corresponds to the non-rotating Drukker-Fiol solution [4].

From Figs. 1, 5 and 6, it is manifestly observed again that the solution with $(C_1, C_2) = (1.2, 0.1)$ is composed of a dual giant graviton propagating along the tunneling trajectory and a spike D3-brane solution. For the values of C_1 and C_2 , the presence of the dual giant graviton is obvious. As C_2 increases or C_1 decreases, the spike tends to be wider compared to the radius of the dual giant graviton and absorbs it.

Thus the solution (3.11) is attached to a circle or a straight line on the Poincaré AdS boundary and it is carrying an angular momentum J from a point on the boundary to another.

Finally we shall give a comment on the operator corresponding to the Euclidean D3-brane solution. As we have already explained, a natural candidate should be a circular or a straight-line Wilson loop in k -th symmetric representation with local operator insertions. However, for the solutions with J of order N or larger, it may be necessary to take account of non-planar contributions for the local operators. That is, we may have to replace Z^J with the dual giant graviton operator like (2.10). In fact, in the dual gravity side, the rotating D3-brane is composed of a dual giant Wilson loop and a dual giant graviton. We will further discuss the corresponding gauge-theory operator again in section 5.

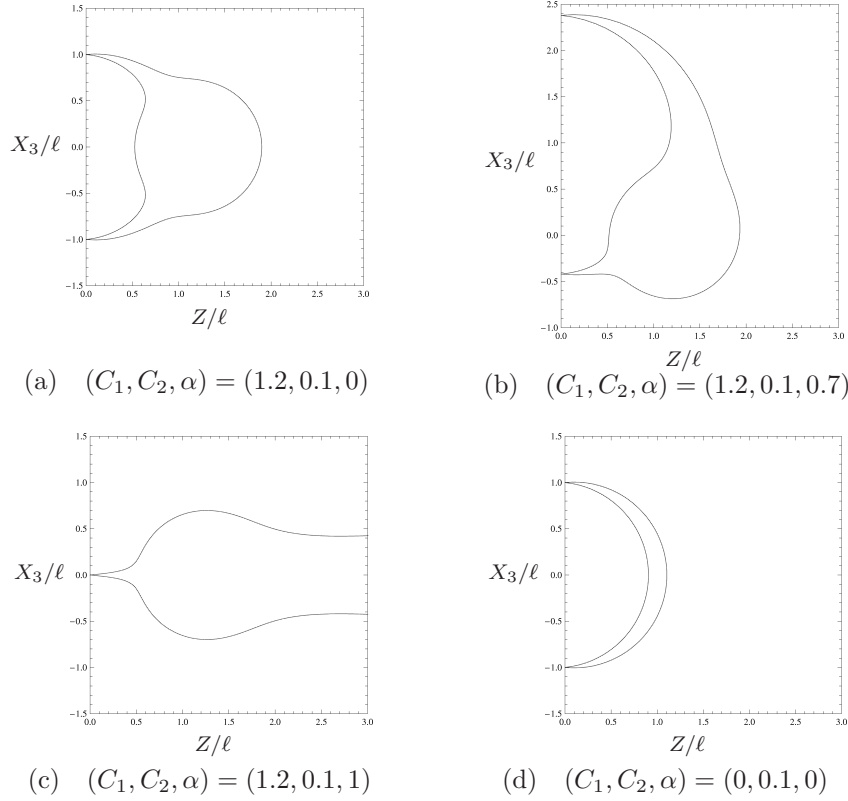


Figure 6: The time slices of Fig. 5 at $t = 0$.

4 Evaluation of D3-brane action

Let us evaluate the classical action of the Euclidean D3-brane solution. For simplicity we omit the subscript “E” hereafter. All t and ϕ in the following should be understood as t_E and ϕ_E .

In addition to the DBI action S_{DBI} and the WZ term S_{WZ} , we have to add appropriate boundary terms to adjust the boundary conditions properly. First of all, it is necessary to consider the usual boundary term for the Legendre transformation of the radial coordinate $u = 1/Z$ of the AdS_5 [28]. Then we have to introduce additional boundary terms for other Legendre transformations because the solution carries the conserved charges: the string charge k and the angular momentum J .

After all, the following summation should be considered as the total action:

$$S_{\text{total}} = S_{\text{DBI}} + S_{\text{WZ}} + S_{\phi} + S_A + S_u. \quad (4.1)$$

The last three terms are the boundary terms for the Legendre transformations with respect to the angle variable ϕ , the gauge potential A and the radial coordinate $u = 1/Z$.

For the solution we consider, all the terms in (4.1) contain divergences. Hence it is necessary

to introduce cutoffs t_{\min} , t_{\max} and ρ_{\max} , and restrict the range of the integration as

$$t_{\min} < t < t_{\max}, \quad \rho_{\min} < \rho < \rho_{\max}, \quad 0 \leq \varphi_1 \leq \pi, \quad 0 \leq \varphi_2 \leq 2\pi.$$

Remember that ρ_{\min} is defined by $\sin \chi(\rho_{\min}) = 1$, or $\rho_{\min} = 0$ for the solution with $\chi(\rho) = 0$.

We use two notations of the world-volume coordinate hereafter. The one is the notation we used so far, and ρ is regarded as a world-volume coordinate. Then we have to consider the two regions $0 \leq \chi \leq \pi/2$ and $\pi/2 \leq \chi \leq \pi$. The other is to use χ as a world-volume coordinate, instead of ρ . Then the whole solution can be covered with a single patch. Hereafter we shall occasionally use this single-patch notation, where the range of the parameter χ is restricted as $\chi_{\min} < \chi < \chi_{\max}$ with $\chi_{\min} = \chi(\rho_{\max})$ and $\chi_{\max} = \pi - \chi(\rho_{\max})$.⁹

From now on we evaluate the α - and ℓ -dependence of each term in (4.1). To make our discussion clear, we shall summarize below the relevant steps of the calculation and the results only. We refer the readers, who are interested in the detailed calculations, to Appendices.

4.1 Evaluation of $S_{\text{DBI}} + S_{\text{WZ}}$

The aim here is to evaluate the contributions coming from the DBI action and the WZ term:

$$\begin{aligned} S_{\text{DBI}} + S_{\text{WZ}} &= \int dt d\rho d\varphi_1 d\varphi_2 \mathcal{L} \\ &= T_{\text{D3}} \int dt d\rho d\varphi_1 d\varphi_2 \sqrt{\det(G_{ab} + 2\pi\alpha' F_{ab})} - T_{\text{D3}} \int \mathcal{P}_\alpha[\mathcal{C}_4]. \end{aligned} \quad (4.2)$$

Recall that there are contributions from two patches, though it is not written down explicitly. Our notation will be explained shortly.

Remember that the solution (3.11) should be derived from the action defined on the double Wick rotated geometry:

$$\frac{ds^2}{L^2} = \frac{dZ^2 + (dX_1)^2 + (dX_2)^2 + (dX_3)^2 + (dX_4)^2}{Z^2} + d\theta^2 - \sin^2 \theta d\phi^2, \quad (4.3)$$

and the imaginary ansatz for the electric flux:

$$F_{t\rho} = -i \frac{L^2}{2\pi\alpha'} F. \quad (4.4)$$

The explicit form of S_{DBI} is

$$S_{\text{DBI}} = 2 \times \frac{2N}{\pi} \int_{t_{\min}}^{t_{\max}} dt \int_{\rho_{\min}}^{\rho_{\max}} d\rho \frac{C_2^3 (\cosh^4 \rho - C_1^2) \cosh^2 \rho}{(\cosh^2 \rho - C_1^2)^2 \sqrt{\cosh^2 \rho - C_1^2 - C_2^2 \coth^2 \rho}}. \quad (4.5)$$

⁹Note that, for the solution with $\chi(\rho) = 0$, we need to take ρ as a world volume coordinate.

The overall factor 2 implies that there are two patches. Possible dependence on α and ℓ arises only through the definition of the cutoffs.

As for the WZ term, there is an ambiguity related to the gauge transformation: Under the gauge transformation the RR potential may change by an exact form, and it may affect the value of the WZ term since boundaries of the D3-brane should be taken into account.

In order to fix this ambiguity, we just follow the proposal of [25] and use the following RR potential,

$$\mathcal{C}_4 = \frac{L^4}{Z^4} dX_4 \wedge dX_1 \wedge dX_2 \wedge dX_3. \quad (4.6)$$

The pull back of the RR potential on the D3-brane solution is represented by $\mathcal{P}_\alpha[\mathcal{C}_4]$ and defined on the space spanned by $(t, \chi, \varphi_1, \varphi_2)$.¹⁰ As it can be seen from (3.11), the pull back depends on α but it is independent of ℓ . The solutions with different values of α are related through coordinate transformation. This is the case for the pull backs with different α , and the WZ term

$$S_{\text{WZ}} = -T_{\text{D3}} \int \mathcal{P}_\alpha[\mathcal{C}_4] \quad (4.7)$$

depends on α through the boundary term.

In order to evaluate the WZ term, it is convenient to introduce a three form Λ_3^α as follows:

$$\mathcal{P}_\alpha[\mathcal{C}_4] - \mathcal{P}[\tilde{\mathcal{C}}_4] = d\Lambda_3^\alpha. \quad (4.8)$$

Here $\mathcal{P}[\tilde{\mathcal{C}}_4]$ is written in terms of $(t, \chi, \varphi_1, \varphi_2)$ as

$$\mathcal{P}[\tilde{\mathcal{C}}_4] = L^4 \sinh^4 \rho(\chi) \sin^2 \chi \sin \varphi_1 dt \wedge d\chi \wedge d\varphi_1 \wedge d\varphi_2, \quad (4.9)$$

where $\rho(\chi)$ is the inverse function of $\chi = \chi(\rho)$ (and also $\chi = \pi - \chi(\rho)$) and the explicit form of Λ_3^α is given in Appendix B. Note that the DBI action (4.5) can be written as¹¹

$$S_{\text{DBI}} = T_{\text{D3}} \int \mathcal{P}[\tilde{\mathcal{C}}_4]. \quad (4.10)$$

By using (4.7), (4.8) and (4.10), we obtain the following expression,

$$S_{\text{DBI}} + S_{\text{WZ}} = -T_{\text{D3}} \int_{\text{b}} \Lambda_3^\alpha. \quad (4.11)$$

Here the subscript “b” implies that the integral is over the boundary of the space parametrized by $(t, \chi, \varphi_1, \varphi_2)$.

¹⁰We use the single-patch notation to discuss the WZ term.

¹¹This just means the fact that $S_{\text{DBI}} + S_{\text{WZ}} = 0$ in the original global coordinate.

The non-vanishing components of Λ_3^α are as follows:

$$\Lambda_3^\alpha = (\Lambda_3^\alpha)_{t\chi\varphi_2} dt \wedge d\chi \wedge d\varphi_2 + (\Lambda_3^\alpha)_{t\varphi_1\varphi_2} dt \wedge d\varphi_1 \wedge d\varphi_2 + (\Lambda_3^\alpha)_{\chi\varphi_1\varphi_2} d\chi \wedge d\varphi_1 \wedge d\varphi_2.$$

The explicit forms are given by (B.11) with (B.4) and (B.6)–(B.10). Then the right-hand side of (4.11) can be rewritten as

$$\begin{aligned} \int_b \Lambda_3^\alpha = & \int_{t_{\min}}^{t_{\max}} dt \int_{\chi_{\min}}^{\chi_{\max}} d\chi \int_0^{2\pi} d\varphi_2 \left[(\Lambda_3^\alpha)_{t\chi\varphi_2} \right]_{\varphi_1=0}^{\varphi_1=\pi} - \int_{t_{\min}}^{t_{\max}} dt \int_0^\pi d\varphi_1 \int_0^{2\pi} d\varphi_2 \left[(\Lambda_3^\alpha)_{t\varphi_1\varphi_2} \right]_{\chi_{\min}}^{\chi_{\max}} \\ & + \int_{\chi_{\min}}^{\chi_{\max}} d\chi \int_0^\pi d\varphi_1 \int_0^{2\pi} d\varphi_2 \left[(\Lambda_3^\alpha)_{\chi\varphi_1\varphi_2} \right]_{t_{\min}}^{t_{\max}}. \end{aligned} \quad (4.12)$$

All the integrands in (4.12) obviously are independent of ℓ . The ℓ -dependence might arise through the cutoffs ρ_{\max} and $t_{\min, \max}$, but the integrals actually converge and do not depend on ℓ . The α -dependence of the integrals is discussed in Appendices B and C. Here we just summarize the results:

1. 1st-term: $(\Lambda_3^\alpha)_{t\chi\varphi_2}$ -term

This term vanishes thanks to the following relation (see (B.4) and (B.6)):

$$(\Lambda_3^\alpha)_{t\chi\varphi_2} \Big|_{\varphi_1=0} = (\Lambda_3^\alpha)_{t\chi\varphi_2} \Big|_{\varphi_1=\pi} = 0.$$

2. 2nd-term: $(\Lambda_3^\alpha)_{t\varphi_1\varphi_2}$ -term

A detailed calculation is summarized in Appendix C.1. Here we rely on a numerical calculation in a step of the integrals. The result is as follows:

$$\begin{aligned} & \int_{t_{\min}}^{t_{\max}} dt \int_0^\pi d\varphi_1 \int_0^{2\pi} d\varphi_2 \left[(\Lambda_3^\alpha)_{t\varphi_1\varphi_2} \right]_{\chi_{\min}}^{\chi_{\max}} \\ & \xrightarrow{|t_{\min, \max}|, \rho_{\max} \rightarrow \infty} \begin{cases} 4\pi^2 L^4 (\operatorname{arcsinh} C_2 - C_2 \sqrt{1 + C_2^2}) & (\alpha = 1) \\ 0 & (\alpha \neq 1) \end{cases}. \end{aligned}$$

3. 3rd-term: $(\Lambda_3^\alpha)_{\chi\varphi_1\varphi_2}$ -term

As explained in Appendix C.2, this term is independent of α in the large $|t_{\min, \max}|$ limit. The integral is performed over the boundary at $t = t_{\min, \max}$ and it is localized near the “points” where the local operators are inserted. Therefore the result should not depend on α , because α describes the global structure of the solution.

In summary, $S_{\text{DBI}} + S_{\text{WZ}}$ is given by

$$S_{\text{DBI}} + S_{\text{WZ}} = \text{const.} + \begin{cases} 2N \left(\operatorname{arcsinh} C_2 - C_2 \sqrt{1 + C_2^2} \right) & (\alpha = 1) \\ 0 & (\alpha \neq 1) \end{cases}. \quad (4.13)$$

Here “const.” is a finite constant independent of α and ℓ , but it may depend on C_1 and C_2 .

4.2 Evaluation of $S_\phi + S_A + S_u$

Next we discuss the boundary terms S_ϕ , S_A and S_u .

The boundary terms S_ϕ , S_A and S_u are defined, respectively, as

$$S_\phi = - \int_{\rho_{\min}}^{\rho_{\max}} d\rho d\varphi_1 d\varphi_2 \left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \phi \right]_{t_{\min}}^{t_{\max}}, \quad (4.14)$$

$$S_A = - \int_{\rho_{\min}}^{\rho_{\max}} d\rho d\varphi_1 d\varphi_2 \left[\frac{\partial \mathcal{L}}{\partial F_{t\rho}} A_\rho \right]_{t_{\min}}^{t_{\max}}, \quad (4.15)$$

$$S_u = - \int_{\rho_{\min}}^{\rho_{\max}} d\rho d\varphi_1 d\varphi_2 \left[\frac{\partial \mathcal{L}}{\partial \dot{u}} u \right]_{t_{\min}}^{t_{\max}} - \int_{t_{\min}}^{t_{\max}} dt d\varphi_1 d\varphi_2 \left. \frac{\partial \mathcal{L}}{\partial u'} u \right|_{\rho_{\max}}. \quad (4.16)$$

There are implicitly contributions from the two patches.

As for (4.14) and (4.15), the dependence on α and ℓ may come only from the cutoffs. By performing ρ -integrals and summing contributions from the two patches, we have the following results:

$$\begin{aligned} S_\phi &= (t_{\max} - t_{\min})|J|, \\ S_A &= 2 \times \frac{2N}{\pi} \int_{t_{\min}}^{t_{\max}} dt \int_{\rho_{\min}}^{\rho_{\max}} d\rho C_2 \frac{\cosh^4 \rho - C_1^2}{\cosh^2 \rho \sqrt{\cosh^2 \rho - C_1^2 - C_2^2 \coth^2 \rho}} \\ &= \frac{4N}{\pi} (t_{\max} - t_{\min}) C_2 \frac{\sqrt{-C_1^2 \sinh^2 \rho_{\max} + \cosh^2 \rho_{\max} (\sinh^2 \rho_{\max} - C_2^2)}}{\cosh \rho_{\max}}. \end{aligned} \quad (4.17)$$

Then let us consider (4.16). It is composed of two terms. The first term gives α - and ℓ -independent contribution as explained in Appendix D.1, while the second term depends on α and ℓ . By using $u = 1/Z$ and substituting the solution, the second term can be rewritten as

$$2\pi L^4 C_2 T_{\text{D3}} \sqrt{\cosh^2 \rho_{\max} - C_1^2 - C_2^2 \coth^2 \rho_{\max}} \int_0^\pi d\varphi_1 \sin \varphi_1 \int_{t_{\min}}^{t_{\max}} dt \left. \frac{Z'}{Z} \right|_{\rho_{\max}}. \quad (4.18)$$

Here Z'/Z is given by

$$\frac{Z'}{Z} = - \frac{A \cosh t + B}{C \cosh t + D},$$

with the symbols A , B , C and D defined, respectively, as

$$\begin{aligned} A &= \sinh \rho, \\ B &= \sqrt{1 - \alpha^2} \cos \varphi_1 (\cosh \rho \sin \chi + \sinh \rho \cos \chi \chi') + \alpha (\cosh \rho \cos \chi - \sinh \rho \sin \chi \chi'), \\ C &= \cosh \rho, \\ D &= \sinh \rho (\sqrt{1 - \alpha^2} \sin \chi \cos \varphi_1 + \alpha \cos \chi), \end{aligned}$$

for the patch with $0 \leq \chi \leq \pi/2$. The terms for the other patch $\pi/2 \leq \chi \leq \pi$ are given by the usual replacement $\chi(\rho) \rightarrow \pi - \chi(\rho)$.

The t -integral can analytically be performed and the result is

$$\int_{t_{\min}}^{t_{\max}} dt \frac{Z'}{Z} = \left[-\frac{A}{C}t - \frac{2(-BC + AD)}{C\sqrt{C^2 - D^2}} \arctan \left(-\sqrt{\frac{C-D}{C+D}} \tanh \left(\frac{t}{2} \right) \right) \right]_{t_{\min}}^{t_{\max}}. \quad (4.19)$$

Then the φ_1 -integral has to be performed. The first term in (4.19) does not depend on φ_1 . Hence the φ_1 -integral can be easily carried out and the result is

$$-2 \times \frac{2N}{\pi} C_2 (t_{\max} - t_{\min}) \frac{\sqrt{-C_1^2 \sinh^2 \rho_{\max} + \cosh^2 \rho_{\max} (\sinh^2 \rho_{\max} - C_2^2)}}{\cosh \rho_{\max}}.$$

The overall factor 2 comes from taking the two patches. This term exactly cancels the boundary term S_A .

The φ_1 -dependence of the second term in (4.19) is a little bit complicated. The φ_1 -integral is evaluated in the large $|t_{\min, \max}|$ and large ρ_{\max} limit in Appendix D.2. The result of φ_1 -integral depends on the patches and also on α . The convergence of the integral again assures that the result does not depend on ℓ , i.e., possible dependences arise only through the cutoffs since Z'/Z is independent of ℓ . For the detailed calculation, see Appendix D.2.

After all, we have shown that

$$S_u + S_A = \text{const.} + \begin{cases} 4NC_2\sqrt{1+C_2^2} & (\alpha = 1) \\ 0 & (\alpha \neq 1) \end{cases}.$$

By gathering the results, $S_\phi + S_A + S_u$ has been evaluated as follows:

$$S_\phi + S_A + S_u = \text{const.} + 2|J| \log \left(\frac{2\ell}{\epsilon} \right) + \begin{cases} 4NC_2\sqrt{1+C_2^2} & (\alpha = 1) \\ 0 & (\alpha \neq 1) \end{cases}. \quad (4.20)$$

The cutoff ϵ is defined as

$$-t_{\min} = t_{\max} \equiv \log(2\ell/\epsilon).$$

This definition is equivalent to the standard cutoff $Z = \epsilon$ imposed at the center $\rho = 0$ of the D3-brane in the equal t slice in terms of the original global coordinates.

4.3 Total action

By gathering (4.13) and (4.20), the total action is given by

$$S_{\text{total}} = f(C_1, C_2) + 2|J| \log \left(\frac{2\ell}{\epsilon} \right) + \begin{cases} 2N(\text{arcsinh } C_2 + C_2\sqrt{1+C_2^2}) & (\alpha = 1) \\ 0 & (\alpha \neq 1) \end{cases}. \quad (4.21)$$

Here $f(C_1, C_2)$ is a function of C_1 and C_2 . This reproduces the result of [22] by setting $k = 1$ and taking large N .

At first sight, it might be curious to find the result of Drukker-Fiol for a circular loop [4] with inverse sign in the third term of a straight line ($\alpha = 1$). But note that the third term itself does not make sense, because it is just a part of the total action. Indeed, no one knows the normalization constant of the gauge-theory operator and hence even the overall normalization of S_{total} does not have any physical significance in our analysis. Remember that this is the case even in the string case [22], where the authors pointed out also a subtlety concerning a regularization in the presence of an R-charge.

A possible resolution proposed in [22] is to take the difference of the total action as

$$S_{\text{total}}|_{\alpha \neq 1} - S_{\text{total}}|_{\alpha = 1}, \quad (4.22)$$

and to compare it with the difference between a circular and a straight-line Wilson loop without R-charge. In fact, the difference (4.22) for our total action (4.21) properly reproduces the result of the Drukker-Fiol. Inversely speaking, this prescription works well even for the D3-brane solution. Thus our result gives a non-trivial support for the proposal in [22].

Consistency with $J \rightarrow 0$ limit

It may be interesting to see that (4.21) is consistent with $J \rightarrow 0$ limit. It is not obvious to check whether (4.21) really reproduces the result of [4]. Here an ingredient of importance is the constant term $f(C_1, C_2)$ in (4.21). In the case with $J = 0$, we need to compute $f(0, C_2)$.¹² It is still too complicated to do analytically, so we have numerically evaluated it. The result supports that

$$f(0, C_2) = -2N(\text{arcsinh } C_2 + C_2\sqrt{1 + C_2^2}). \quad (4.23)$$

Thus (4.21) reduces to

$$S_{\text{total}} = \begin{cases} 0 & (\alpha = 1) \\ -2N(\text{arcsinh } C_2 + C_2\sqrt{1 + C_2^2}) & (\alpha \neq 1) \end{cases}. \quad (4.24)$$

This is nothing but the result of [4]. The mechanism to reproduce (4.24) is somewhat non-trivial, because it reappears from different integrals. The problem for the normalization of the Wilson loop might be clarified by investigating the behavior of $f(C_1, C_2)$ more in detail.

¹²More precisely, it is necessary to evaluate (C.5) and (D.2) with $C_1 = 0$. In particular, in computing (C.5), the only contribution comes from the first term in (B.10) at $t = t_{\text{max}}$.

Interpretation of ℓ dependence

The ℓ -dependence of (4.21) should come from the contraction of local operators inserted in the loop. Then they have to have conformal dimension J due to the agreement of R-charge. This ℓ -dependence is consistent with the expectation value of the Wilson loop with the insertions of Z^J and its complex conjugate [19].

However, the identification of [19] should be modified to realize the fact that the solution is composed of the dual giant Wilson loop and a dual giant graviton rather than a BPS particle. A key observation is that the ℓ -dependence is also consistent with the propagator of dual giant gravitons. We will propose another candidate of the dual gauge-theory operator for the D3-brane solution in the next section.

5 What is the corresponding Wilson loop?

Finally let us discuss the Wilson loop corresponding to the D3-brane solution. The classical action computed in the previous section should be an important key to identify it.

The local operator inserted in the loop should be modified by taking account of the non-planar contributions. It is reasonable to consider a dual giant graviton operator Z_M^N as an inserted operator. Here M and N are the indices of J -th symmetric representation and its conjugate representation.

Thus a plausible candidate for the gauge-theory operator corresponding to the Euclidean D3-brane solution would possibly be the following:

$$\Gamma_{BN}^{CM} \tilde{\Gamma}_{DQ}^{AP} [\mathcal{W}_{\vec{X}_f}^{\vec{X}_i}(C)]_A^B [Z(\vec{X}_i)]_M^N [\mathcal{W}_{\vec{X}_i}^{\vec{X}_f}(C)]_C^D [\bar{Z}(\vec{X}_f)]_P^Q. \quad (5.1)$$

Here $[\mathcal{W}_{\vec{X}_i}^{\vec{X}_f}(C)]_A^B$ represents the k -th symmetric Wilson line running from \vec{X}_i to \vec{X}_f along the loop C . At $\vec{X} = \vec{X}_i$ and \vec{X}_f , there are four indices (B, C, N, M) and (D, A, Q, P) , respectively. For the gauge invariance, these indices must be contracted separately at each of the points with some coefficients Γ_{BN}^{CM} and $\tilde{\Gamma}_{DQ}^{AP}$.

A simple way to contract the indices may be taking $\Gamma_{BN}^{CM} = \delta_B^C \delta_N^M$ and $\tilde{\Gamma}_{DQ}^{AP} = \delta_D^A \delta_Q^P$. Then the operator (5.1) just reduces to:

$$\text{Tr}_{S_k} \mathcal{W}(C) \text{Tr}_{S_J} Z(\vec{X}_i) \text{Tr}_{S_J} \bar{Z}(\vec{X}_f). \quad (5.2)$$

This is just the multiplication of Wilson loop without local operator insertions with standard dual giant graviton operators, and it does not reduce to (2.14) when $k = 1$ and $J \ll N$.

An example of the operator which reduces to (2.14) can be constructed by combining k -th and J -th symmetric indices into a $(k+J)$ -th symmetric one. We explain this type of operator by expressing the symmetric indices in terms of fundamental indices as:

$$\begin{aligned} Z_M^N &= Z_{\{m_1, \dots, m_J\}}^{\{n_1, \dots, n_J\}} \equiv S_{m_1, \dots, m_J}^{\bar{m}_1, \dots, \bar{m}_J} S_{\bar{n}_1, \dots, \bar{n}_J}^{n_1, \dots, n_J} Z_{\bar{m}_1}^{\bar{n}_1} \cdots Z_{\bar{m}_J}^{\bar{n}_J}, \\ \mathcal{W}_A^B &= \mathcal{W}_{\{a_1, \dots, a_k\}}^{\{b_1, \dots, b_k\}} \equiv S_{a_1, \dots, a_k}^{\bar{a}_1, \dots, \bar{a}_k} S_{\bar{b}_1, \dots, \bar{b}_k}^{b_1, \dots, b_k} \mathcal{W}_{\bar{a}_1}^{\bar{b}_1} \cdots \mathcal{W}_{\bar{a}_k}^{\bar{b}_k}. \end{aligned}$$

All the lower-case indices express the (anti-) fundamental indices and the tensor $S_{m_1, \dots, m_J}^{n_1, \dots, n_J}$ is totally symmetric with respect to the upper (or lower) indices. In this notation, the operator in which k -th and J -th symmetric indices are combined to $(k+J)$ -th symmetric indices can be written down as

$$S_{b_1, \dots, b_k, n_1, \dots, n_J}^{c_1, \dots, c_k, m_1, \dots, m_J} S_{d_1, \dots, d_k, q_1, \dots, q_J}^{a_1, \dots, a_k, p_1, \dots, p_J} [\mathcal{W}_{\vec{X}_f}^{\vec{X}_i}]_{\{a_1, \dots, a_k\}}^{\{b_1, \dots, b_k\}} [Z(\vec{X}_i)]_{\{m_1, \dots, m_J\}}^{\{n_1, \dots, n_J\}} [\mathcal{W}_{\vec{X}_f}^{\vec{X}_i}]_{\{c_1, \dots, c_k\}}^{\{d_1, \dots, d_k\}} [\bar{Z}(\vec{X}_f)]_{\{p_1, \dots, p_J\}}^{\{q_1, \dots, q_J\}}.$$

In the case with $k=1$, this operator can be written as (up to a normalization constant)

$$\text{Tr}_F \left[\mathcal{W}_{\vec{X}_f}^{\vec{X}_i}(C) Z(X_i)^J \mathcal{W}_{\vec{X}_i}^{\vec{X}_f}(C) \bar{Z}(\vec{X}_f)^J \right] + \text{multi-trace operators}, \quad (5.3)$$

and it reduces to the operator (2.14) when we assume $J \ll N$, since the multi-trace operators become sub-leading in $1/N$.

By assuming other coefficients Γ_{BN}^{CM} and $\tilde{\Gamma}_{DQ}^{AP}$, we can consider more generic Wilson loops with local operator insertions which seem to be consistent with the expectation value predicted by the D3-brane action. It would be nice to seek the definite choice of the coefficients Γ_{BN}^{CM} and $\tilde{\Gamma}_{DQ}^{AP}$, for example, via perturbative computation in the gauge-theory side. We leave this issue as a future work.

6 Conclusion and discussion

We have reexamined a rotating D3-brane solution in Lorentzian signature [25] and discussed its tunneling picture.

We first observed that the solution is composed of a Drukker-Fiol solution [4] and a dual giant graviton [10, 11]. From this observation, we argued that the corresponding operator is a k -th symmetric Wilson loop with dual giant graviton operator insertions.

Then we have performed a double Wick rotation for the solution by following the prescription of [20] and constructed the solution in Euclidean AdS. For this solution, the total classical action including appropriate boundary terms has been evaluated. The resulting action reproduces the expectation value of the k -th symmetric Wilson loop without local operator insertions and the logarithmically divergent term which is consistent with the correlation function of the dual giant graviton operators.

These results may suggest that the usual prescription to compute the expectation value of Wilson loop by using the Gaussian matrix model can also be applied to the present case. It would be nice to find further supports in confirmation of the dual operator (5.1). It is also necessary to have an argument to fix the coefficients of (5.1). We leave these issues as future works.

It would also be interesting to try to construct a rotating D5-brane. In the case of giant Wilson loop the shape of D5-brane is $\text{AdS}_2 \times S^4$. Hence the S^4 part is expanding in S^5 and so it seems difficult to find an appropriate ansatz in the same way as the case of dual giant Wilson loop. However, from our observation given in this paper, we can easily guess that the desired solution should be composed of an $\text{AdS}_2 \times S^4$ D5-brane (giant Wilson loop) and a giant graviton. By considering a giant spike solution [29] and deforming it, it may be possible to find a rotating giant Wilson loop. The corresponding Wilson loop in the gauge-theory side should be obvious. All we have to do is to replace the k -th symmetric representation and the dual giant graviton operator with the k -th anti-symmetric representation and the giant graviton operator.

Furthermore it may be possible to construct a solution composed of dual giant Wilson loop and giant graviton, or of giant Wilson loop and dual giant graviton. It is nice to try to find such a solution.

It is also nice to study quantum fluctuations around the rotating D3-brane solution. The fluctuations around the string solution of [22] have been discussed in [30]. The resulting action is very complicated. But we can clearly see the asymptotic behavior of the Lagrangian around the boundary and at the center of AdS. It behaves as the semiclassical action around an AdS_2 solution [31]¹³ around the boundary, while as the pp-wave string at the center of AdS. The similar behavior should be expected even for the fluctuations around the D3-brane solution.

We hope that the D-brane dynamics discussed in this paper would be an important key to clarify some dynamical aspects of (dual) giant Wilson loops.

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¹³For semiclassical approximation of DBI actions around AdS-branes see [32].

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Appendix

Hereafter we omit the subscript “E” of t_E and set $L = 1$ for simplicity.

A Coordinate transformation

The following decomposition of the coordinate transformation (2.8) will be used in the next appendix:

1. Change from global coordinates to the Poincaré coordinates:

$$z = \frac{e^t}{\cosh \rho}, \quad r = e^t \tanh \rho, \quad (\text{A.1})$$

$$x_1 = r \sin \chi \sin \varphi_1 \cos \varphi_2, \quad x_2 = r \sin \chi \sin \varphi_1 \sin \varphi_2, \quad (\text{A.2})$$

$$x_3 = r \sin \chi \cos \varphi_1, \quad x_4 = r \cos \chi. \quad (\text{A.3})$$

2. Rotation in the (x_3, x_4) -space:

$$\begin{pmatrix} x'_3 \\ x'_4 \end{pmatrix} = \begin{pmatrix} \alpha & -\sqrt{1-\alpha^2} \\ \sqrt{1-\alpha^2} & \alpha \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}, \quad x'_{1,2} = x_{1,2}, \quad z' = z. \quad (\text{A.4})$$

3. Translation into the x'_4 -direction:

$$x''_4 = x'_4 + 1, \quad x''_{1,2,3} = x'_{1,2,3}, \quad z'' = z'. \quad (\text{A.5})$$

4. Inversion transformation and sign flip of x''_4 :

$$Z'' = \frac{z''}{(x''_i)^2 + z''^2}, \quad X''_{1,2,3} = \frac{x''_{1,2,3}}{(x''_i)^2 + z''^2}, \quad X''_4 = -\frac{x''_4}{(x''_i)^2 + z''^2}. \quad (\text{A.6})$$

5. Translation into the X''_4 -direction:

$$X'_4 = X''_4 + \frac{1}{2}, \quad X'_{1,2,3} = X''_{1,2,3}, \quad Z' = Z''. \quad (\text{A.7})$$

6. Scale transformation in five dimensions:

$$Z = 2\ell Z', \quad X_i = 2\ell X'_i. \quad (\text{A.8})$$

B Derivation of the boundary three form Λ_3^α

We shall derive the explicit form of Λ_3^α in (4.8). For this purpose it is convenient to consider \mathcal{C}_4 in (4.6) as the α -dependent four-form on the space spanned by $(t, \rho, \chi, \varphi_1, \varphi_2)$ by using (2.8):

$$\mathcal{C}_4[Z, X_i] = \mathcal{C}_4^\alpha[t, \rho, \chi, \varphi_1, \varphi_2].$$

Let us consider Λ_4^α defined as

$$\Lambda_4^\alpha[t, \rho, \chi, \varphi_2, \varphi_2] \equiv \mathcal{C}_4^\alpha[t, \rho, \chi, \varphi_1, \varphi_2] - \tilde{\mathcal{C}}_4[t, \rho, \chi, \varphi_1, \varphi_2].$$

Here $\tilde{\mathcal{C}}_4[t, \rho, \chi, \varphi_1, \varphi_2]$ is defined as

$$\tilde{\mathcal{C}}_4[t, \rho, \chi, \varphi_1, \varphi_2] = \sinh^4 \rho \sin^2 \chi \sin \varphi_1 \left(dt + \frac{d\rho}{\sinh \rho \cosh \rho} \right) \wedge d\chi \wedge d\varphi_1 \wedge d\varphi_2. \quad (\text{B.1})$$

By setting $\rho = \rho(\chi)$, it reduces to (4.9). With Λ_4^α , $d\Lambda_3^\alpha$ in (4.8) is rewritten as

$$d\Lambda_3^\alpha = \Lambda_4^\alpha[t, \rho = \rho(\chi), \chi, \varphi_1, \varphi_2].$$

First it is easy to check that $\tilde{\mathcal{C}}_4[t, \rho, \chi, \varphi_1, \varphi_2]$ is rewritten as

$$\tilde{\mathcal{C}}_4[t, \rho, \chi, \varphi_1, \varphi_2] = \frac{1}{z^4} dx_4 \wedge dx_1 \wedge dx_2 \wedge dx_3. \quad (\text{B.2})$$

Here (z, x_i) are related to $(t, \rho, \chi, \varphi_1, \varphi_2)$ via (A.1)–(A.3). Since the steps 2., 3., 5., and 6. in the previous subsection keep the form of the four form potentials (4.6) and (B.2), we have

$$\begin{aligned} \Lambda_4^\alpha &\equiv \frac{1}{Z^4} dX_4 \wedge dX_1 \wedge dX_2 \wedge dX_3 - \frac{1}{z^4} dx_4 \wedge dx_1 \wedge dx_2 \wedge dx_3 \\ &= \frac{1}{Z''^4} dX_4'' \wedge dX_1'' \wedge dX_2'' \wedge dX_3'' - \frac{1}{z''^4} dx_4'' \wedge dx_1'' \wedge dx_2'' \wedge dx_3''. \end{aligned}$$

Here (Z'', X_i'') and (z'', x_i'') are related to $(t, \rho, \chi, \varphi_1, \varphi_2)$ via (A.1)–(A.6). With help of (A.6) we can write Λ_4^α in terms of (z'', x_i'') as:

$$\begin{aligned} \Lambda_4^\alpha &= \frac{2}{z''^4} \frac{1}{g} (z'' dz'' \wedge \Delta_3 - z''^2 \Omega_4), \\ \Delta_3 &= (x_4'' dx_1'' - x_1'' dx_4'') \wedge dx_2'' \wedge dx_3'' + (x_2'' dx_3'' - x_3'' dx_2'') \wedge dx_4'' \wedge dx_1'', \\ \Omega_4 &= dx_4'' \wedge dx_1'' \wedge dx_2'' \wedge dx_3'', \quad g = z''^2 + (x_i'')^2. \end{aligned} \quad (\text{B.3})$$

We further introduce the polar coordinates $(r'', \theta_1, \theta_2, \theta_3)$ defined as

$$\begin{aligned} x_1'' &= r'' \sin \theta_1 \sin \theta_2 \cos \theta_3, & x_2'' &= r'' \sin \theta_1 \sin \theta_2 \sin \theta_3, \\ x_3'' &= r'' \sin \theta_1 \cos \theta_2, & x_4'' &= r'' \cos \theta_1. \end{aligned}$$

By using them, Δ_3 and Ω_4 can be rewritten into the following forms:

$$\begin{aligned}\Delta_3 &= r''^4 \sin^2 \theta_1 \sin \theta_2 d\theta_1 \wedge d\theta_2 \wedge d\theta_3, \\ \Omega_4 &= r''^3 \sin^2 \theta_1 \sin \theta_2 dr'' \wedge d\theta_1 \wedge d\theta_2 \wedge d\theta_3.\end{aligned}$$

Then Λ_4^α is also rewritten as the exact form:

$$\Lambda_4^\alpha = d\tilde{\Lambda}_3^\alpha, \quad \tilde{\Lambda}_3^\alpha = \left(-\frac{r''^2}{z''^2} + \log \left(1 + \frac{r''^2}{z''^2} \right) \right) \frac{\Delta_3}{r''^4}. \quad (\text{B.4})$$

With (A.1)–(A.5), $\tilde{\Lambda}_3^\alpha$ and Δ_3 can be expressed in terms of $(t, \rho, \chi, \varphi_1, \varphi_2)$. Then z'' and r'' can be rewritten as

$$z''^2 = \frac{e^{2t}}{\cosh^2 \rho}, \quad r''^2 = e^{2t} \tanh^2 \rho + 2e^t \tanh \rho (\alpha \cos \chi + \sqrt{1 - \alpha^2} \sin \chi \cos \varphi_1) + 1. \quad (\text{B.5})$$

Now Δ_3 is given by

$$\begin{aligned}\Delta_3 &= (\Delta_3)_{t\chi\varphi_2} dt \wedge d\chi \wedge d\varphi_2 + (\Delta_3)_{\rho\chi\varphi_2} d\rho \wedge d\chi \wedge d\varphi_2 + (\Delta_3)_{t\varphi_1\varphi_2} dt \wedge d\varphi_1 \wedge d\varphi_2 \\ &\quad + (\Delta_3)_{\rho\varphi_1\varphi_2} d\rho \wedge d\varphi_1 \wedge d\varphi_2 + (\Delta_3)_{\chi\varphi_1\varphi_2} d\chi \wedge d\varphi_1 \wedge d\varphi_2,\end{aligned}$$

where the non-vanishing components are

$$(\Delta_3)_{t\chi\varphi_2} = -\sqrt{1 - \alpha^2} e^{3t} \tanh^3 \rho \sin \chi \sin^2 \varphi_1, \quad (\text{B.6})$$

$$(\Delta_3)_{\rho\chi\varphi_2} = -\sqrt{1 - \alpha^2} e^{3t} \frac{\tanh^2 \rho}{\cosh^2 \rho} \sin \chi \sin^2 \varphi_1, \quad (\text{B.7})$$

$$(\Delta_3)_{t\varphi_1\varphi_2} = \alpha e^{3t} \tanh^3 \rho \sin^3 \chi \sin \varphi_1 - \sqrt{1 - \alpha^2} e^{3t} \tanh^3 \rho \sin^2 \chi \cos \chi \sin \varphi_1 \cos \varphi_1, \quad (\text{B.8})$$

$$(\Delta_3)_{\rho\varphi_1\varphi_2} = \alpha e^{3t} \frac{\tanh^2 \rho}{\cosh^2 \rho} \sin^3 \chi \sin \varphi_1 - \sqrt{1 - \alpha^2} e^{3t} \frac{\tanh^2 \rho}{\cosh^2 \rho} \sin^2 \chi \cos \chi \sin \varphi_1 \cos \varphi_1, \quad (\text{B.9})$$

$$\begin{aligned}(\Delta_3)_{\chi\varphi_1\varphi_2} &= e^{4t} \tanh^4 \rho \sin^2 \chi \sin \varphi_1 + \alpha e^{3t} \tanh^3 \rho \sin^2 \chi \cos \chi \sin \varphi_1 \\ &\quad + \sqrt{1 - \alpha^2} e^{3t} \tanh^3 \rho \sin^3 \chi \sin \varphi_1 \cos \varphi_1. \quad (\text{B.10})\end{aligned}$$

Finally Λ_3^α is given by

$$\Lambda_3^\alpha[t, \chi, \varphi_1, \varphi_2] = \tilde{\Lambda}_3^\alpha[t, \rho = \rho(\chi), \chi, \varphi_1, \varphi_2]. \quad (\text{B.11})$$

C Integral of $(\Lambda_3^\alpha)_{t\varphi_1\varphi_2}$ and $(\Lambda_3^\alpha)_{\chi\varphi_1\varphi_2}$

C.1 $(\Lambda_3^\alpha)_{t\varphi_1\varphi_2}$

The aim here is to perform the integral,

$$\int_{t_{\min}}^{t_{\max}} dt \int_0^\pi d\varphi_1 \int_0^{2\pi} d\varphi_2 \left[(\Lambda_3^\alpha)_{t\varphi_1\varphi_2} \right]_{\chi_{\min}}^{\chi_{\max}}. \quad (\text{C.1})$$

The explicit form of the integrand is given by (B.11) with (B.4) and (B.8). For $\alpha \neq 1$, this integral vanishes in the limit $\rho_{\max} \rightarrow \infty$ or equivalently in the limit $\chi_{\min} \rightarrow 0$ and $\chi_{\max} \rightarrow \pi$. This can be shown by rewriting the integral (C.1) in the form in which a single $\sin \chi$ is extracted as an overall factor, i.e., in the form as $(\sin \chi) \times \int dt(\dots)$. In this form we can show that the t -integral still converges. Since the extracted overall factor $\sin \chi$ vanishes in the limit $\rho_{\max} \rightarrow \infty$, (C.1) vanishes.

However the case with $\alpha = 1$ is special and then r'' at $t = 0$ is given by

$$r''^2 = \tanh^2 \rho + 2 \tanh \rho \cos \chi + 1 \quad (\text{at } t = 0). \quad (\text{C.2})$$

Therefore r'' becomes zero at the upper boundary $\chi = \chi_{\max}$ taking the limit $\chi_{\max} \rightarrow \pi$. We can also check that $r''^2/z''^2 \rightarrow C_2^2$ in the same limit. This means that, for finite C_2 , the integrand of (C.1) tends to diverge at $t = 0$ in the limit $\chi_{\max} \rightarrow \pi$. This divergence cancels the small overall factor $\sin \chi$ and hence the integral (C.1) may give a finite value to S_{WZ} . Nevertheless most terms in the integral actually vanish apart from the following two integrals:

1.

$$\begin{aligned} & \int dt d\varphi_1 d\varphi_2 \left[-\frac{1}{z''^2 r''^2} e^{3t} \tanh^3 \rho \sin^3 \chi \sin \varphi_1 \right]_{\chi_{\min}}^{\chi_{\max}} \\ &= -4\pi \int_{t_{\min}}^{t_{\max}} dt \left[\frac{e^t \cosh^2 \rho \tanh^3 \rho \sin^3 \chi}{e^{2t} \tanh^2 \rho + 2e^t \tanh \rho \cos \chi + 1} \right]_{\chi_{\min}}^{\chi_{\max}} \\ &= 4\pi \left[\sinh^2 \rho \sin^2 \chi \left(\arctan \left[\frac{1 + \tanh \rho \cos \chi}{\tanh \rho \sin \chi} \right] + \arctan \left[\frac{\tanh \rho + \cos \chi}{\sin \chi} \right] \right. \right. \\ &\quad \left. \left. - \arctan \left[\frac{\tanh \rho \cos \chi + e^{-t_{\min}}}{\tanh \rho \sin \chi} \right] - \arctan \left[\frac{\cos \chi + e^{t_{\max}} \tanh \rho}{\sin \chi} \right] \right) \right]_{\chi_{\min}}^{\chi_{\max}} \\ &\rightarrow 4\pi C_2^2 \left[\left(0 + 0 - \frac{\pi}{2} - \frac{\pi}{2} \right) - \left(\frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} \right) \right] \quad (|t_{\min, \max}|, \rho_{\max} \rightarrow \infty) \quad (\text{C.3}) \\ &= -4\pi^2 C_2^2. \end{aligned}$$

In (C.3), among two contributions $-\pi$ and 0 in the large round bracket, the first one, i.e., $-\pi$, is the contribution from $\chi = \chi_{\max}$ and the second vanishing term is from the lower boundary $\chi = \chi_{\min}$.

2.

$$\begin{aligned} & \int dt d\varphi_1 d\varphi_2 \left[\frac{1}{r''^4} \log \left(1 + \frac{r''^2}{z''^2} \right) e^{3t} \tanh^3 \rho \sin^3 \chi \sin \varphi_1 \right]_{\chi_{\min}}^{\chi_{\max}} \\ &= 4\pi \left[\tanh^3 \rho \sin^2 \chi \int dt \sin \chi e^t \frac{-t + 2 \log(\cosh \rho) + \log(e^t + e^{-t} + 2 \tanh \rho \cos \chi)}{(e^t \tanh^2 \rho + 2 \tanh \rho \cos \chi + e^{-t})^2} \right]_{\chi_{\min}}^{\chi_{\max}}. \quad (\text{C.4}) \end{aligned}$$

On the lower boundary $\chi = \chi_{\min}$, the integrand develops no singularity in the limit $\chi_{\min} \rightarrow 0$. As for the behavior of the integrand at the boundary $t \sim t_{\min}, t_{\max}$ of the domain of integration, we have

$$\text{the integrand of (C.4)} \rightarrow \begin{cases} e^{-t_{\max}} \frac{2 \log(\cosh \rho_{\max})}{\tanh^4 \rho_{\max}} \sin \chi_{\min} \\ -2t_{\min} e^{3t_{\min}} \sin \chi_{\min} \end{cases}.$$

Hence the integral is finite in the large ρ_{\max} and the large $|t_{\min, \max}|$ limit. On the other hand, the extra overall factor becomes zero in this limit:

$$\tanh^3 \rho_{\max} \sin^2 \chi_{\min} \rightarrow 0.$$

Thus we have

$$\text{the lower boundary contribution of (C.4)} \rightarrow 0.$$

At the upper boundary $\chi = \chi_{\max}$, by the numerical analysis, we found that

$$\text{the upper boundary contribution of (C.4)} \rightarrow 4\pi^2 (C_2^2 - C_2 \sqrt{1 + C_2^2} + \text{arcsinh} C_2).$$

In summary, we obtain that

$$(C.4) \rightarrow 4\pi^2 (-C_2 \sqrt{1 + C_2^2} + \text{arcsinh} C_2).$$

C.2 $(\Lambda_3^\alpha)_{\chi\varphi_1\varphi_2}$

The aim here is to evaluate the α - and ℓ -dependence of the integral:

$$\int_{\chi_{\min}}^{\chi_{\max}} d\chi \int_0^\pi d\varphi_1 \int_0^{2\pi} d\varphi_2 \left[(\Lambda_3^\alpha)_{\chi\varphi_1\varphi_2} \right]_{t_{\min}}^{t_{\max}}. \quad (C.5)$$

The integrand is given by (B.11) with (B.4), (B.9) and (B.10). Although the integrand depends only on α , ℓ -dependence may arise through the cutoffs.

Let us examine each of the terms in the integrand. First, the integrals of the terms in (B.9) and (B.10) which are proportional to e^{3t} vanish in the limit $|t_{\min, \max}| \rightarrow \infty$. This is because the integrals of these terms can be rewritten as $e^{-|t_{\min, \max}|} \times \int d\chi(\dots)$ in which the χ -integral gives no divergence.

Next let us consider the term linear in e^{4t} in (B.10) whose power is greater than the previous case by e^t . Although the contribution from the lower edge, $t = t_{\min}$, vanishes, there may be non-trivial contribution from the upper edge, $t = t_{\max}$. It is easy to check the convergence of the integral in the limit $t_{\max} \rightarrow \infty$ and $\rho_{\max} \rightarrow \infty$. What is more we can also check that the contribution does not depend on α . This is essentially because the α -dependence is

sub-leading with respect to $e^{-t_{\max}}$, as can be seen from the expression of r'' in (B.5).¹⁴ Hence we understand that the integral (C.5) does not depend on α nor on ℓ in the large $|t_{\min, \max}|$ and ρ_{\max} limit.

D Evaluation of S_u

The boundary term S_u is composed of the two terms like

$$S_u = \int_{\rho_{\min}}^{\rho_{\max}} d\rho d\varphi_1 d\varphi_2 \left[\frac{\partial \mathcal{L}}{\partial \dot{Z}} Z \right]_{t_{\min}}^{t_{\max}} + \int_{t_{\min}}^{t_{\max}} dt d\varphi_1 d\varphi_2 \frac{\partial \mathcal{L}}{\partial Z'} Z \Big|_{\rho_{\max}}. \quad (\text{D.1})$$

D.1 ρ -integral

It is easy to check that the large $|t_{\min, \max}|$ and ρ_{\max} limit of the first term in (D.1) converges to give the following expression:

$$-2J - \frac{8N}{\pi} C_2 \int d\rho \frac{(\cosh^2 \rho - C_1^2)^2 + C_2^2 \cosh^4 \rho}{\cosh^2 \rho (\cosh^2 \rho - C_1^2) \sqrt{\cosh^2 \rho - C_1^2 - C_2^2 \coth^2 \rho}}. \quad (\text{D.2})$$

Here we have summed contributions from two patches. From this expression, it is clear that the first term of (D.1) does not depend on α and ℓ . In the case with $k \sim 1$ and $C_1 = 1$, (D.2) is reduced to the result of [22] as

$$-2J - \frac{2k\sqrt{\lambda}}{\pi}. \quad (\text{D.3})$$

D.2 t -integral

The t -integral of (D.1) is given by (4.18)–(4.19). In the main text, we have performed the φ_1 -integral of the first term of (4.19). We consider here the second term. In the large $|t_{\min, \max}|$ and large ρ_{\max} limit, the second term of (4.19) will be estimated as

1. $\alpha = 1$

$$\begin{aligned} -BC + AD &\rightarrow \begin{cases} -1 - C_2^2 & (\chi = \chi_{\min}) \\ 1 + C_2^2 & (\chi = \chi_{\max}) \end{cases}, \\ C\sqrt{C^2 - D^2} &\rightarrow \begin{cases} \cosh \rho_{\max} \sqrt{1 + C_2^2} & (\chi = \chi_{\min}) \\ \cosh \rho_{\max} \sqrt{1 + C_2^2} & (\chi = \chi_{\max}) \end{cases}, \end{aligned}$$

¹⁴In the case with $\rho_{\min} = 0$, we need to consider ρ -integral instead of χ -integral. Then for the range $\rho \sim 0$ the same argument can not be applied since r'' can vanish. However, the α -independence can easily be checked even for the range by Taylor expanding the log term in (B.4).

$$\sqrt{\frac{C-D}{C+D}} \tanh\left(\frac{t}{2}\right) \rightarrow \begin{cases} 0 & (\chi = \chi_{\min}, t = t_{\max}) \\ 0 & (\chi = \chi_{\min}, t = t_{\min}) \\ +\infty & (\chi = \chi_{\max}, t = t_{\max}) \\ -\infty & (\chi = \chi_{\max}, t = t_{\min}) \end{cases}.$$

The integrand of φ_1 -integral becomes as follows:

$$\begin{cases} 0 & (\chi = \chi_{\min}, t = t_{\max}) \\ 0 & (\chi = \chi_{\min}, t = t_{\min}) \\ NC_2\sqrt{1+C_2^2}\sin\varphi_1 & (\chi = \chi_{\max}, t = t_{\max}) \\ -NC_2\sqrt{1+C_2^2}\sin\varphi_1 & (\chi = \chi_{\max}, t = t_{\min}) \end{cases}.$$

2. $\alpha \neq 1$

$$\begin{aligned} -BC + AD &\rightarrow \begin{cases} C_2\sqrt{1-\alpha^2}\cos\varphi_1\cosh\rho_{\max} & (\chi = \chi_{\min}) \\ C_2\sqrt{1-\alpha^2}\cos\varphi_1\cosh\rho_{\max} & (\chi = \chi_{\max}) \end{cases}, \\ C\sqrt{C^2-D^2} &\rightarrow \begin{cases} \sqrt{1-\alpha^2}\cosh^2\rho_{\max} & (\chi = \chi_{\min}) \\ \sqrt{1-\alpha^2}\cosh^2\rho_{\max} & (\chi = \chi_{\max}) \end{cases}, \\ \sqrt{\frac{C-D}{C+D}} \tanh\left(\frac{t}{2}\right) &\rightarrow \begin{cases} \sqrt{\frac{1-\alpha}{1+\alpha}} & (\chi = \chi_{\min}, t = t_{\max}) \\ -\sqrt{\frac{1-\alpha}{1+\alpha}} & (\chi = \chi_{\min}, t = t_{\min}) \\ \sqrt{\frac{1+\alpha}{1-\alpha}} & (\chi = \chi_{\max}, t = t_{\max}) \\ -\sqrt{\frac{1+\alpha}{1-\alpha}} & (\chi = \chi_{\max}, t = t_{\min}) \end{cases}. \end{aligned}$$

The integrand of φ_1 -integral becomes as follows:

$$\begin{cases} (2N/\pi)C_2^2\sin\varphi_1\cos\varphi_1\arctan\left(\sqrt{\frac{1-\alpha}{1+\alpha}}\right) & (\chi = \chi_{\min}, t = t_{\max}) \\ -(2N/\pi)C_2^2\sin\varphi_1\cos\varphi_1\arctan\left(\sqrt{\frac{1-\alpha}{1+\alpha}}\right) & (\chi = \chi_{\min}, t = t_{\min}) \\ (2N/\pi)C_2^2\sin\varphi_1\cos\varphi_1\arctan\left(\sqrt{\frac{1+\alpha}{1-\alpha}}\right) & (\chi = \chi_{\max}, t = t_{\max}) \\ -(2N/\pi)C_2^2\sin\varphi_1\cos\varphi_1\arctan\left(\sqrt{\frac{1+\alpha}{1-\alpha}}\right) & (\chi = \chi_{\max}, t = t_{\min}) \end{cases}.$$

By performing φ_1 -integral, we have

1. $\alpha = 1$:

$$\int_{t_{\min}}^{t_{\max}} dt d\varphi_1 d\varphi_2 \left. \frac{\partial \mathcal{L}}{\partial Z'} Z \right|_{\chi_{\max}} = -\frac{1}{2} S_A + 4NC_2\sqrt{1+C_2^2}, \quad \int_{t_{\min}}^{t_{\max}} dt d\varphi_1 d\varphi_2 \left. \frac{\partial \mathcal{L}}{\partial Z'} Z \right|_{\chi_{\min}} = -\frac{1}{2} S_A.$$

2. $\alpha \neq 1$:

$$\int_{t_{\min}}^{t_{\max}} dt d\varphi_1 d\varphi_2 \left. \frac{\partial \mathcal{L}}{\partial Z'} Z \right|_{\chi_{\max}} = \int_{t_{\min}}^{t_{\max}} dt d\varphi_1 d\varphi_2 \left. \frac{\partial \mathcal{L}}{\partial Z'} Z \right|_{\chi_{\min}} = -\frac{1}{2} S_A.$$

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